# SPLINE-BASED APPROACH TO OPTIMAL CONTROL OF TRAJECTORIES UNDER INEQUALITY TYPE CONSTRAINTS

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**Abstract.** The paper is devoted to an optimal trajectory planning problem considered as a problem of constrained optimal control for dynamical systems. It is one of the fundamental problems in robotics, biomechanics, aeronautics and many other areas of application of control theory. The simplest version of this problem supposes that there are given sequences of target points and prescribed times, and we are required to be at the given point at the prescribed time. However, in most of the applications, it is enough when the trajectory passes close to the assigned point at the prescribed time. So, the location conditions could be considered as the inequality type constraints. The aim of this research is to reduce such an optimal control problem to the problem of splines in convex sets, which could be analysed and solved by methods of the general theory of splines. Dynamical systems associated with the second order linear differential equation with initial conditions are investigated in the paper (the restriction on the order of equations is not essential). We consider this system as a curve generator. The goal is to find a control law by minimization of the quadratic cost function under inequality type constraints on location conditions. A spline-based numerical scheme for some cases of such optimal control problems is proposed in this paper. In particular, the method of adding-removing spline interpolation knots is applied to the construction of its solution. The suggested technique is illustrated by numerical examples.

**Keywords:** splines in convex sets; optimal control; linear dynamical system; trajectory planning.

## Introduction

This paper is devoted to the trajectory planning as a special case of problems of optimal control, i.e., determining control and state trajectories for a linear dynamical system over a period of time to minimize an objective functional. The dynamical system under consideration is realized as a curve generator. The goal is to find a control law which will drive the output trajectory in such a way that constraints on location conditions are satisfied.

The trajectory planning problem is a fundamental problem in many fields connected with robots, mechanics and so on (see, e.g., [1-6]). Usually this problem consists of constructing a function of time that satisfies initial conditions together with other requirements such as via points, obstacle avoidance or reflection of the dynamics of the considered system. When via points are specified, trajectories may be constructed as interpolating curves to pass the via points or as approximating curves to pass near the points (see, e.g., [7; 8]). A version of the problem when we are required to be at the point at the fixed time is typical for such problems as path planning in air traffic control and many problems in industrial robotics because in the industrial applications such as cutting, welding, assembling and so on, the robots are strongly required to follow a desired trajectory. However, in the most of applications, the requirement to pass through specific points at specific times is overly restrictive, it is enough when we go reasonably close to these given points. As it is noted in [2], these types of relaxed interpolation problems exhibit properties that are desirable for two different reasons. First of all, small deviations from the waypoints can result in a significant decrease in the cost and secondly, when the data that we work with, is noise contaminated, it is even not desirable to interpolate through these points precisely. For example, in applications in the field of robotics the time of passage at via points influences not only the kinematic properties of the motion, but also the dynamic ones (see specifications in, e.g., [1]). All these restrictions can affect rejection of exact interpolating conditions. Typically, some deviations from the exact location are allowed even in air traffic control. As an example, passenger comfort requires that accelerations are minimized and that transitions are smooth, so we cannot require the exact interpolation in this situation. The problems with some allowed deviations from the given points or the problems of the obstacle avoidance trajectory planning are often treated by introducing a cost nonlinear function consisting of the distance to obstacles and some others characteristics of the trajectory. Such approach is closely related to the idea of smoothing in the theory of splines.

Classical polynomial splines are usually used as interpolating splines; that means, they are required to pass through target points at prescribed times. Smoothing splines, in contrast, are only required to pass “close” to the data points. Specifically, the smoothing spline model is a smooth function $s$ from a suitable function space that minimizes the objective functional with a parameter of a weight which can be interpreted as a compromise between the smoothing and the closeness of fit to the prescribed values. From the point of view of control systems, the smoothing spline model is closely related to the finite horizon linear quadratic optimal control problem by treating derivatives of $s$ as a control input (see, e.g., [2, 9]). This fact has led to a highly interesting spline model defined by a linear control system called a control theoretic spline. Control theoretic splines are introduced as a generalization of smoothing splines (see, e.g., [10]). It is shown in [2] and the references therein that a number of smoothing, interpolation and trajectory planning problems can be incorporated into control problems and studied using control theory and optimization techniques on Hilbert spaces with efficient numerical schemes. Control theoretic splines give a richer class of smoothing curves relative to polynomial curves. They have been proved to be useful for the construction of solutions in the problems of trajectory planning of robotic arms or mobile robots in service applications for human guiding and assistance, and to approximate the contour of encountered obstacles in the environment, to the problem of contour modelling or reconstruction and so on (see, e.g., [7; 11-15]). It should also be mentioned that in most of the papers about control splines (see, e.g., [2; 9; 12]) the problems on splines are reduced to the problem of the control theory. The aim of this paper is, on the contrary, to transform the problem of the control theory into the corresponding problem of smoothing splines with the aim to use for its analysis methods and results of the general theory of splines.

There are various types of optimal control problems, depending on the performance index, the type of time domain (continuous, discrete), the presence of different types of constraints and possibilities to choose some free variables. In this paper we consider a dynamical system associated to the second order linear differential equation with initial conditions. In general, it can be stated as

$$x'(t) = Mx(t) + \beta u(t), \quad y(t) = \gamma^T x(t), \quad t \in [a, b], \tag{1}$$

considered with the initial condition

$$x(a) = c. \tag{2}$$

Here $x$ is a vector-valued absolutely continuous function defined on $[a, b]$, $M$ is a given quadratic constant matrix and $\beta$, $\gamma$ are given constant vectors of compatible dimensions. We consider system (1)-(2) as the curve $z = y(t)$ generator. The goal is to find a control law $u \in L_2[a, b]$ which drives the scalar output trajectory close to a sequence of set points at fixed times

$$\{(t_i, z_i): i = 1, 2, \ldots, n\}, \text{ where } a < t_1 < t_2 < \ldots < t_n \le b, \tag{3}$$

by minimization of the objective functional

$$\int_a^b u^2(t)dt\,. \tag{4}$$

In the case of trajectory planning, when we need to generate curves that pass exactly through via points, we refer to the classical statement of the problem under consideration:

$$\int_a^b u^2(t)dt \to \min_{u\in L_2[a,\, b]:\;\; (1)-(2) \;\text{ and }\; y(t_i)=z_i,\; i=1,\ldots,n}, \tag{5}$$

but our main interest is to consider the problem when the trajectory passes close to a predefined point at the prescribed time. So the location conditions could be considered as the inequality type constraints, for example, by using the nonnegative parameters $\varepsilon_i$, $i = 1,\ldots,n$:

$$\int_a^b u^2(t)dt \to \min_{u\in L_2[a,\, b]:\;\; (1)-(2) \;\text{ and }\; |y(t_i)-z_i|\le\varepsilon_i,\; i=1,\ldots,n}. \tag{6}$$

In this paper problem (1)-(2) is considered in the case of

$$M = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

So dynamical system (1) can be expressed as the second order linear differential equation

$$f''(t) + pf'(t) + qf(t) = u(t),$$

here $f$ is used to denote $x_1$. For this case problem (6) can be rewritten as

$$\int_a^b \left( f''(t) + pf'(t) + qf(t) \right)^2 dt \to \min_{\substack{f \in L_2[a,\, b]:\ f(a)=c_1,\ f'(a)=c_2, \\ |y(t_i)-z_i| \le \varepsilon_i,\ i=1,\dots,n}}, \tag{7}$$

where $y(t) = \gamma_1 f(t) + \gamma_2 f'(t)$.

## Spline-Based Approach

Problem (7) corresponds to the following more general conditional minimization problem:

$$\|Tf\|_{L_2[a,\, b]} \to \min_{\substack{f \in L_2^r[a,\, b]:\ (Af)_0 = c_1,\ (Af)_{n+1} = c_2, \\ |(Af)_i - z_i| \le \varepsilon_i,\ i=1,\dots,n}}, \tag{8}$$

where linear operators $T:L_2^r[a,b] \to L_2[a,b]$ and $A:L_2^r[a,b] \to \mathbb{R}^{n+2}$ are continuous, $L_2^r[a,b]$ is the Sobolev space, vectors $c \in \mathbb{R}^2$, $z = (z_i)_{i=1,\dots,n}$ and $\varepsilon = (\varepsilon_i)_{i=1,\dots,n}$ with $\varepsilon_i \ge 0$, $i = 1,\dots, n$ are given. We assume that $A(L_2^r[a,b]) = \mathbb{R}^{n+2}$. In the case under consideration $r = 2$ and

$$Tf = f''(t) + pf'(t) + qf(t) = u, \quad (Af)_i = \gamma_1 f(t_i) + \gamma_2 f'(t_i),\ i = 1,\dots,n,$$

$$(Af)_0 = f(a), \quad (Af)_{n+1} = f'(a). \tag{9}$$

Problem (8) in the case $\varepsilon_i = 0$, $i = 1,\dots, n$, corresponds to the interpolating problem. By the well-known theorems (see, e.g., Theorems 4.4.2. and 4.5.9. in [16]) its solution is a spline from the space

$$S(T,\, A) = \{ s \in L_2^r[a,\, b] \mid \forall x \in kerA \quad < Ts, Tx > = 0 \}.$$

Here and in the sequel $kerA$ is the kernel of operator $A$ and the corresponding inner product is denoted by $< \cdot , \cdot >$. The form of splines from $S(T, A)$ depending on the parameters $p$ and $q$ for the considered case (9) is obtained in [17] by using the general theorem (see Theorem 1 in [18]) and applying functional analysis tools.

Problem (8) in the case $\varepsilon_i > 0$, $i = 1,\dots, n$, corresponds to the problem (8) on splines in a convex set. The conditions for the existence and uniqueness of solution of (8) in this case follow from the next theorem (it is a special case of Theorem 7 in [18]).

**Theorem.** Under the assumption that $kerT$ is finite-dimensional a solution of problem (8) with $\varepsilon_i \ge 0$, $i = 1,\dots, n$, exists. An element $s \in L_2^r[a,b]$, such as $(As)_0 = c_1$, $(As)_{n+1} = c_2$, $z_i - \varepsilon_i \le (As)_{\le} z_i + \varepsilon_i$, $i = 1,\dots, n$, is a solution of (10) if and only if there exists the vector $\lambda \in \mathbb{R}^{n+2}$ such that

$$T \cdot T_s = A \cdot \lambda \tag{10}$$

and components $\lambda_i$, $i = 1,\dots, n$, satisfy the conditions

$$\begin{aligned} &\lambda_i = 0, \quad \text{if} \quad z_i - \varepsilon_i < (As)_i < z_i + \varepsilon_i, \\ &\lambda_i \ge 0, \quad \text{if} \quad (As)_i = z_i - \varepsilon_i, \\ &\lambda_i \le 0, \quad \text{if} \quad (As)_i = z_i + \varepsilon_i. \end{aligned} \tag{11}$$

Under the additional assumption $kerT \cap kerA = \{0\}$ this solution is unique.

This result implies that a solution of problem (8) in this case belongs to $S(T, A)$. To find it we can use the method of adding-removing interpolation knots which is considered in details, for example, in [19] or [20]. It is an iterative method. We start with a solution $s^1$ obtained by using only the initial conditions and denote $I^1 = \varnothing$. The set of indices $I^k \subset \{1,\dots, n\}$ for interpolation knots $t_i$ and numbers $d_i^k$ are specified during iterations. On the $k$-th step we need to solve the following interpolation problem: to construct a spline $s^k \in S(T, A)$ such that the initial conditions $(As^k)_0 = c_1$, $(As^k)_{n+1} = c_2$ and interpolation conditions written in the form $(As^k)_i = d_i^k$, $i \in I^k$, are satisfied. The iterative step from $I^k$ to $I^{k+1}$ is done by adding to $I^k$ all indices $i \in \{1,..., n\}$ such that the restriction $z_i - \varepsilon_i \leq (As)_i \leq z_i + \varepsilon_i$ is not satisfied. For the added index $i$ we take $d_i^{k+1} = z_i - \varepsilon_i$ if $z_i - \varepsilon_i > (As^k)_i$ and $d_i^{k+1} = z_i + \varepsilon_i$ if $z_i + \varepsilon_i > (As^k)_i$. Additionally, we remove from $I^k$ all indices $i \in I^k$ such that the rule (12) is not satisfied for the corresponding coefficient of $s^k$. To finish the $k$-th step we also denote $d_i^{k+1} = d_i^k$ for $I^{k+1} \cap I^k$. If $I^{k+1} = I^k$ then the algorithm ends and the obtained $s^k$ is a solution of (8).

We note that a similar method for the construction of a solution of the problem of optimal control under fuzzy conditions was considered in [21], but its application to the problem under consideration is technically simpler.

**Numerical Example**

We consider problem (7) as (8) with operator $T$ and $A$ defined by (9). The solution of it belongs to the corresponding $S(T, A)$. The form of splines from $S(T, A)$ is obtained in [17] and it depends on the roots $r_1$, $r_2$ of the equation $r^2 + pr + q = 0$. The following classification has been done in [17]:

- Class 1 (exponential splines with polynomial coefficients): $r_1 = r_2 \in \mathbb{R}\backslash\{0\}$;
- Class 2 (exponential splines): $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$;
- Class 3 (polynomial-exponential splines): $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$, $r_1 \neq 0$, $r_2 = 0$;
- Class 4 (polynomial splines): $r_1 = r_2 = 0$;
- Class 5 (trigonometric splines with polynomial coefficients): $r_{1,2} = \pm i\eta \neq 0$;
- Class 6 (trigonometric splines with exponential-polynomial coefficients): $r_{1,2} = \zeta \pm i\eta$ with $\eta \neq 0$ and $\zeta \neq 0$.

The following numerical example is considered to illustrate the proposed technique. The numerical results are obtained by Maple.

*Example*. We consider the numerical example for the problem (7) with $p = -2$, $q = 1$, $\gamma_1 = 1$ and $\gamma_2 = 0$, interval $[a, b] = [0, 4]$, the initial conditions are with $c_1 = 6$, $c_2 = 0$ and desired points are (1, 1.5), (1.5, 2.8), (2, 4), (3, 1), (4, 5.5), so $n = 5$. The allowed deviations from these points are given by $\varepsilon_i = 0.5$, $i = 1,\dots n$.

These values of $p$ and $q$ correspond to $r_1 = r_2 = 1$, i.e. the solution of (7) is exponential spline with polynomial coefficients in the following form, as it is obtained in [17]:

$$s(t) = (\mu_1 + \mu_2(t-a))e^{r_1(t-a)} + \frac{\lambda_0}{4r_1^3}(r_1(t-a)(e^{r_1(a-t)} + e^{r_1(t-a)}) + e^{r_1(a-t)} - e^{r_1(t-a)}) + \frac{\lambda_{n+1}(t-a)}{4r_1}(e^{r_1(a-t)} - e^{r_1(t-a)}) +$$

$$+\sum_{i=1}^{n}\lambda_i\left(\frac{\gamma_1(t-t_i)_+^0}{4r_1^3}(r_1(t-t_i)(e^{r_1(t_i-t)} + e^{r_1(t-t_i)}) + e^{r_1(t_i-t)} - e^{r_1(t-t_i)}) + \frac{\gamma_2}{4r_1}(e^{r_1(t_i-t)} - e^{r_1(t-t_i)})(t-t_i)_+^0\right). \quad (12)$$

which coefficients fulfil the following system

$$(\gamma_1 + \gamma_2 r_1)\sum_{i=1}^{n}\lambda_i e^{r_1 t_i} + (\lambda_0 + \lambda_{n+1} r_1)e^{r_1 a} = 0,$$

$$(\lambda_0 a + \lambda_{n+1}(ar_1 + 1))e^{r_1 a} + \sum_{i=1}^{n}\lambda_i((\gamma_1 + \gamma_2 r_1)t_i + \gamma_2)e^{r_1 t_i} = 0,$$

but the system of the interpolating conditions for $s(t_i)$ are precised by the iterations of the method of adding-removing knots. The coefficients of solution with the form (12) in the considered case are $\lambda_0 = 60.57700$, $\lambda_1 = -74.94558$, $\lambda_2 = 0$, $\lambda_3 = 59.82526$, $\lambda_4 = -27.97635$, $\lambda_5 = 4.53229$, $\lambda_6 = 15.55907$.

The corresponding control function $u$ is given by

$$u(t) = \sum_{i=1}^{n}\lambda_i e^{r_1(t_i-t)}(t_i-t)_+^0((\gamma_1 + \gamma_2 r_1)(t_i - t) + \gamma_2).$$

We note that in our case point (1.5, 2.8) does not affect our solution $s(t)$ due to the results of the method of adding-removing interpolation knots (we obtain $\lambda_2 = 0$).

To compare our result to the one in the case when there is the requirement to pass the trajectory through specific points we also consider the interpolation version of our problem (7), i.e., when $\varepsilon_i = 0$, $i = 1,\dots n$. The solution of this interpolation problem is exponential spline with polynomial coefficients (12) with the coefficients $\lambda_0 = 73.17760$, $\lambda_1 = -114.69439$, $\lambda_2 = 30.33115$, $\lambda_3 = 63.22836$, $\lambda_4 = -29.31013$, $\lambda_5 = 3.73303$, $\lambda_6 = 20.35442$.

The comparison of the objective functional for the solutions of the problem with inequality type constraints ($\|u(t)\| = 20.0912$) and the interpolation problem ($\|u(t)\| = 68.8480$) proves that the rejection of exact interpolating conditions implies the decreasing in the value of the cost functional.

The corresponding graphs of the solution of (7) in form (12) for both considered cases and the control function are shown in Fig. 1 and Fig. 2.

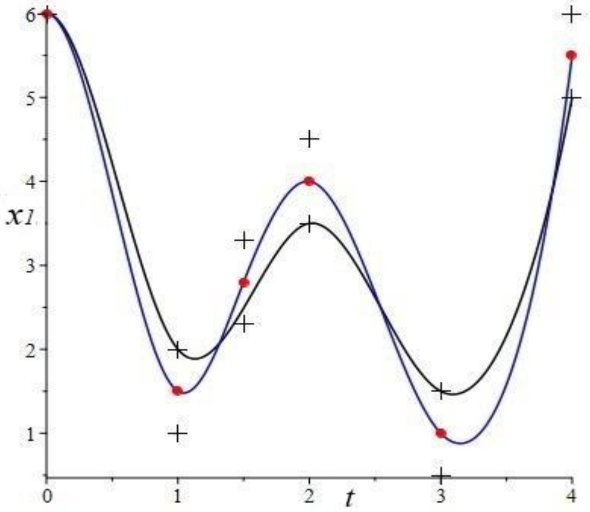


Fig. 1. **State trajectories for *Example*** (black line for the solution of the problem with inequality type constraints, blue line for the interpolating case, red points indicate via points, plus ( + ) signs define the possible deviattions)

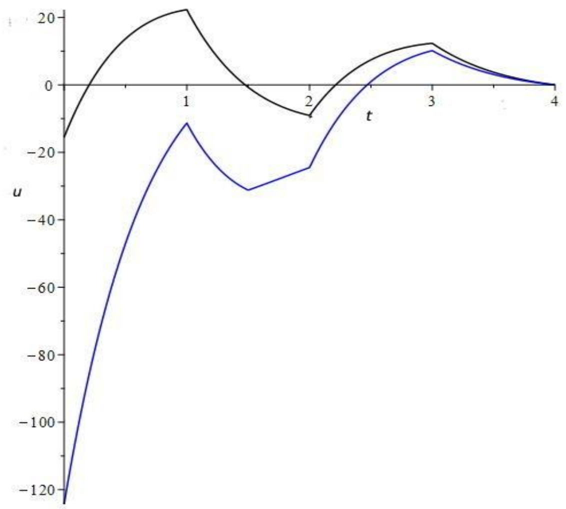


Fig. 2. **Control low for *Example*** (black line for the solution of the problem with inequality type constraints, blue line for the interpolating case)

## Conclusions

The paper is devoted to the spline-based method which allows to analyse the optimal control problem with initial condition by reducing it to the problem of smoothing splines. The method is proposed for the special case of the optimal control problem when a dynamical system is associated with the second order linear differential equation. The suggested technique could also be used for optimal control problems with dynamical systems, which can be reduced to linear differential equations of the order greater than the second.

## Author contributions

Conceptualization and methodology, S.Asmuss and N.Budkina; software, N. Budkina; validation, S. Asmuss; investigation, S. Asmuss and N. Budkina; writing – original draft preparation, S. Asmuss and N. Budkina; writing – review and editing, S. Asmuss and N. Budkina. All authors have read and agreed to the published version of the manuscript.

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